

# QUASICONFORMAL MAPPINGS ON THE HEISENBERG GROUP: AN OVERVIEW

IOANNIS D. PLATIS

ABSTRACT. We present a brief overview of the Korányi-Reimann theory of quasiconformal mappings on the Heisenberg group stressing on the analogies as well as on the differences between the Heisenberg group case and the classical two-dimensional case. We examine the extensions of the theory to more general spaces and we state some known results and open problems.

## CONTENTS

1. Introduction	1
2. Heisenberg group	3
3. A Brief Overview of the Korányi-Reimann Theory	6
4. Further Developments and Some Open Problems	11
References	14

## 1. INTRODUCTION

The classical theory of quasiconformal mappings on the plane was developed first in the Euclidean space  $\mathbb{R}^n$  and produced a variety of results, most of them closely connected to topics in Analysis. In the non-Riemannian setting, the foundations of the theory are tracked down in the works of Mostow and Pansu. In his celebrated 1968 rigidity result [29], Mostow proved that in dimensions  $n > 2$  diffeomorphic compact Riemannian manifolds with constant negative curvature are isometric, in particular they are conformally equivalent. The proof of this result relies on the use of quasiconformal mappings of  $\mathbb{R}^n$ . Later, see [30], he extended this result to the setting of symmetric spaces of rank 1 of non-compact type, i.e., hyperbolic spaces  $\mathbf{H}_{\mathbb{K}}^n$  where  $\mathbb{K}$  can be the set of real numbers  $\mathbb{R}$  (except when  $n = 2$ ), the set of complex numbers  $\mathbb{C}$ , the set of quaternions  $\mathbb{H}$ , or the set of octonions  $\mathbb{O}$  (the latter only when  $n = 2$ ). To obtain this, Mostow had to develop quasiconformal mappings on the boundary of these spaces as an indispensable tool. In rough lines, Mostow's proof in the case  $\mathbb{K} = \mathbb{C}$  and  $n = 2$  goes as follows. Let  $G$  and  $G'$  be two cocompact lattices, i.e.,  $M = \mathbf{H}_{\mathbb{C}}^2/G$  and  $M' = \mathbf{H}_{\mathbb{C}}^2/G'$  are compact and suppose that  $\rho : G \rightarrow G'$  is an isomorphism. From  $\rho$  it is possible to define a quasi-isometric self map  $F$  of  $\mathbf{H}_{\mathbb{C}}^2$  which is equivariant; this map need not be even continuous but has the property that it takes geodesics to quasi-geodesics. Due to a fundamental result in Gromov hyperbolic spaces, from this property  $F$  is extended to a boundary map  $F_{\infty} : \partial\mathbf{H}_{\mathbb{C}}^2 \rightarrow \partial\mathbf{H}_{\mathbb{C}}^2$  which is in fact a quasiconformal homeomorphism of  $\partial\mathbf{H}_{\mathbb{C}}^2$  with respect to a metric comparable

---

*Key words and phrases.* Quasiconformal mappings, Heisenberg group, Korányi-Reimann theory.  
*2010 Mathematics Subject Classification:* 30L10, 30C75.

with the Korányi-Cygan metric. The latter is defined on the (first) Heisenberg group  $\mathfrak{H}_{\mathbb{C}}^1$ ; this group is the  $N$  group in the  $KAN$  decomposition of the symmetric space  $\mathbf{H}_{\mathbb{C}}^2$  and  $\partial\mathbf{H}_{\mathbb{C}}^2$  is the one point compactification of  $\mathfrak{H}$ . After showing that  $F_{\infty}$  has enough regularity, Mostow proves that a  $(G, G')$ -equivariant quasiconformal self map of the boundary is associated with the action of an element of the isometry group  $SU(2, 1)$  of  $\mathbf{H}_{\mathbb{C}}^2 = SU(2, 1)/SU(2)$  and the proof is concluded from the equivariance of the resulting isometry. We refer the interested reader to [9], pp. 135–140, for a short but more detailed description of Mostow’s proof. Pansu obtained a stronger rigidity statement for the cases  $\mathbb{K} = \mathbb{H}$  and  $\mathbb{K} = \mathbb{O}$  in [32]. By using Mostow’s methods, Pansu proved the following property which does not hold for real and complex hyperbolic spaces: every quasi-isometry of quaternionic or octonionic spaces has bounded distance from an isometry. Using the conformal geometry of the boundary which is modelled on the nilpotent group  $\mathfrak{H}_{\mathbb{K}}^{n-1}$  in the  $KAN$  decomposition of the symmetric space  $\mathbf{H}_{\mathbb{K}}^n$  endowed with a Carnot-Carathéodory metric (i.e., a Carnot group), and general properties of Loewner spaces, (for the definitions of Carnot groups and Loewner spaces, see Section 4), he proved that *any* quasiconformal (in fact quasisymmetric) homeomorphism of  $\mathfrak{H}_{\mathbb{K}}^{n-1}$  is actually conformal. This result does not apply to the case  $\mathbb{K} = \mathbb{C}$  and it is an open problem to understand the intrinsic reason for this phenomenon.

Mostow’s rigidity had serious consequences; perhaps two of them are the most significant: First, the moduli space of hyperbolic metrics on a surface  $\Sigma$ , i.e., the Teichmüller space  $\mathcal{T}(\Sigma)$ , which is the case  $\mathbb{K} = \mathbb{R}$  and  $n = 2$  in the above setting, is just a counterexample of rigidity. The proof fails there, since it involves absolute continuity in measure of the boundary quasiconformal (actually quasisymmetric) mappings: in  $S^1 = \partial\mathbf{H}_{\mathbb{R}}^2$  this does not hold.

Second, the theory of quasiconformal mappings on the Heisenberg group emerged, after the pioneering articles of Korányi-Reimann, [22] and [23], and Pansu, [32]. These works constituted a complete framework for the theory of quasiconformal mappings on the Heisenberg group  $\mathfrak{H}^{n-1} := \mathfrak{H}_{\mathbb{C}}^{n-1}$ . Especially, the exposition of Korányi and Reimann is strongly influenced by the then state-of-the-art concerning the quasiconformal mappings in  $\mathbb{R}^n$ . There are many occasions where quasiconformal mappings on the Heisenberg group behave in the same way as those of  $\mathbb{R}^n$ . On the other hand, there are significant differences: For instance, in Euclidean spaces ( $n > 2$ ) there is no Beltrami equation whereas a system of Beltrami equations appears in the Heisenberg group case for all  $n$ . However, in contrast to the real two-dimensional case where there exist solutions to the Beltrami equation, the solvability of the Beltrami system in the Heisenberg group case is still an open problem. According to Korányi and Reimann, even in the case of a full solution of the Beltrami system this is not likely to produce startling results similar to the ones of the plane case. Regularity issues for the quasiconformal maps on the Heisenberg group were first overlooked by Mostow in his original rigidity result. It turned out that the basic property of absolute continuity on lines was much more difficult to obtain than in the Euclidean case. Korányi and Reimann brought this into Mostow’s attention and together they remedied the problem; this correction appears in [23]. Crucial to the further development of the theory was Pansu’s differentiability theorem; as in the Euclidean case, this theorem is derived from the Rademacher-Stepanov Theorem. Pansu’s notion of differentiability turned out to be completely adapted to the structure on the Heisenberg group; the Pansu derivative is founded in such a way that it preserves the grading of the Lie algebra.

All the above ignited the research of quasiconformal mappings in various other spaces; only some of them are Carnot groups, sub-Riemannian manifolds, metric spaces with controlled geometry, etc.

In what follows we will briefly survey the theory of quasiconformal mappings of the first Heisenberg group  $\mathfrak{H}$ , that is,  $\mathfrak{H}_{\mathbb{C}}^1$ . In Section 2 we define the Heisenberg group  $\mathfrak{H}$  and comment on its structures and properties. As the title of Section 3 indicates, we overview there some basic results of the Korányi-Reimann theory. Finally, in Section 4 further developments of the theory are briefly described. We finally underline here that this last section by no means contains all the developments of the theory; it rather reflects to the interests of the author.

## 2. HEISENBERG GROUP

The (first) Heisenberg group  $\mathfrak{H}$  is the set  $\mathbb{C} \times \mathbb{R}$  with multiplication  $*$  given by

$$(z, t) * (w, s) = (z + w, t + s + 2\Im(z\bar{w})),$$

for every  $(z, t)$  and  $(w, s)$  in  $\mathfrak{H}$ . We consider two metrics defined on  $\mathfrak{H}$ ; the first one is induced via the Korányi map  $\alpha : \mathfrak{H} \rightarrow \mathbb{C}$  which is given for every  $(z, t) \in \mathfrak{H}$  by

$$\alpha(z, t) = -|z|^2 + it.$$

Now, the Korányi gauge  $|\cdot|_{\mathfrak{H}}$  is defined by

$$|(z, t)|_{\mathfrak{H}} = |\alpha(z, t)|^{1/2} = ||z|^2 - it|^{1/2},$$

for every  $(z, t) \in \mathfrak{H}$ . Then the *Korányi-Cygan* (or *Heisenberg*) metric  $d_{\mathfrak{H}}$  is defined by the relation

$$d_{\mathfrak{H}}((z_1, t_1), (z_2, t_2)) = |(z_1, t_1)^{-1} * (z_2, t_2)|.$$

Note that the  $d_{\mathfrak{H}}$ -sphere of radius  $R > 0$  and centered at the origin, called the *Korányi sphere*, is

$$\mathcal{S}_{\mathfrak{H}}(R) = \{(z, t) \in \mathfrak{H} \mid |(z, t)|_{\mathfrak{H}} = R\}.$$

The metric  $d_{\mathfrak{H}}$  is invariant under:

- (1) *Left translations*  $T_{(\zeta, s)}$ ,  $(\zeta, s) \in \mathfrak{H}$ , that is, left actions of  $\mathfrak{H}$  onto itself, which are given by

$$T_{(\zeta, s)}(z, t) = (\zeta, s) * (z, t),$$

for every  $(z, t) \in \mathfrak{H}$ ;

- (2) *rotations*  $R_{\theta}$ ,  $\theta \in \mathbb{R}$ , around the vertical axis  $\mathcal{V} = \{0\} \times \mathbb{R}$ , which are given by

$$R_{\theta}(z, t) = (ze^{i\theta}, t),$$

for every  $(z, t) \in \mathfrak{H}$ .

Left translations are left actions of  $\mathfrak{H}$  onto itself and rotations are induced by an action of  $U(1)$  on  $\mathfrak{H}$ ; together they form the group  $\text{Isom}^+(\mathfrak{H}, d_{\mathfrak{H}})$  of *orientation-preserving Heisenberg isometries*. The full group  $\text{Isom}(\mathfrak{H}, d_{\mathfrak{H}})$  of *Heisenberg isometries* comprises compositions of elements of  $\text{Isom}^+(\mathfrak{H}, d_{\mathfrak{H}})$  with the

- 3) *conjugation*  $J$  which is defined by

$$J(z, t) = (\bar{z}, -t),$$

for every  $(z, t) \in \mathfrak{H}$ .

We also consider two other kinds of transformations, namely:

4) *Dilations*  $D_\delta$ ,  $\delta > 0$ . These are defined by

$$D_\delta(z, t) = (\delta z, \delta^2 t),$$

for every  $(z, t) \in \mathfrak{H}$ ;

5) *inversion*  $I$ , which is defined in  $\mathfrak{H} \setminus \{(0, 0)\}$  by

$$I(z, t) = \left( z (\alpha(z, t))^{-1}, -t |\alpha(z, t)|_{\mathfrak{H}}^{-2} \right).$$

The metric  $d_{\mathfrak{H}}$  is scaled up to multiplicative constants by the action of dilations. Also, for each  $p, q \in \mathfrak{H} \setminus \{o = (0, 0)\}$  we have

$$d_{\mathfrak{H}}(I(p), I(q)) = \frac{d_{\mathfrak{H}}(p, q)}{d_{\mathfrak{H}}(o, p) \cdot d_{\mathfrak{H}}(o, q)}.$$

Compositions of orientation-preserving Heisenberg isometries, dilations and inversion form the *similarity group*  $\text{Sim}(\mathfrak{H})$  of  $\mathfrak{H}$ . Recall the *complex hyperbolic plane*  $\mathbf{H}_{\mathbb{C}}^2$ : That is, the symmetric space  $\text{SU}(2, 1)/\text{SU}(2)$ . The Heisenberg group  $\mathfrak{H}$  is the  $N$  group in its  $KAN$  decomposition and the boundary  $\partial\mathbf{H}_{\mathbb{C}}^2$  is the one point compactification of  $\mathfrak{H}$ , that is,  $\partial\mathbf{H}_{\mathbb{C}}^2 = \mathfrak{H} \cup \{\infty\}$ . It can be proved that the action of  $\text{SU}(2, 1) = \text{Isom}(\mathbf{H}_{\mathbb{C}}^2)$  on the boundary  $\partial\mathbf{H}_{\mathbb{C}}^2 = \mathfrak{H} \cup \{\infty\}$  is completely described by the transformations (1)-(5), that is, if  $g$  is an isometry, then it is the composition of transformations of the form (1)-(5).

The Heisenberg group  $\mathfrak{H}$  is a two-step nilpotent Lie group with underlying manifold  $\mathbb{C} \times \mathbb{R}$ ; its left translations are  $T_{(\zeta, s)}$ ,  $(\zeta, s) \in \mathfrak{H}$ , as in (1). Let  $(z = x + iy, t)$  be the coordinates of  $\mathfrak{H}$  and consider the left invariant vector fields

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

and the complex fields

$$Z = \frac{1}{2}(X - iY) = \frac{\partial}{\partial z} + i\bar{z} \frac{\partial}{\partial t}, \quad \bar{Z} = \frac{1}{2}(X + iY) = \frac{\partial}{\partial \bar{z}} - iz \frac{\partial}{\partial t}.$$

The Lie algebra of left invariant vector fields of  $\mathfrak{H}$  has a grading  $\mathfrak{h} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$  with

$$\mathfrak{v}_1 = \text{span}_{\mathbb{R}}\{X, Y\} \quad \text{and} \quad \mathfrak{v}_2 = \text{span}_{\mathbb{R}}\{T\}.$$

The only non-zero bracket relation is

$$[X, Y] = 4T.$$

The Heisenberg group constitutes the prototype of both contact and sub-Riemannian geometry. The contact form  $\omega$  of  $\mathfrak{H}$  is defined as the unique 1-form satisfying  $X, Y \in \ker \omega$ ,  $\omega(T) = 1$ . Uniqueness here is modulo change of coordinates as it follows by the contact version of Darboux's Theorem. The distribution in  $\mathfrak{H}$  defined by the first layer  $\mathfrak{v}_1 := \text{H}(\mathfrak{H})$  is called the *horizontal distribution*. Explicitly, in the Heisenberg coordinates  $z = x + iy, t$ , we have

$$\omega = dt + 2(xdy - ydx) = dt + 2i\Im(\bar{z}dz).$$

The *sub-Riemannian metric*  $\langle \cdot, \cdot \rangle$  is defined on  $\text{H}(\mathfrak{H})$  by the relations

$$\langle X, X \rangle = \langle Y, Y \rangle = 1, \quad \langle X, Y \rangle = \langle Y, X \rangle = 0.$$

The corresponding norm shall be denoted by  $\|\cdot\|$ . The *Legendrian foliation* of  $\mathfrak{H}$  is the foliation of  $\mathfrak{H}$  by horizontal curves. An absolutely continuous curve  $\gamma : [a, b] \rightarrow \mathfrak{H}$  (in the Euclidean sense) with

$$\gamma(\tau) = (\gamma_h(\tau), \gamma_3(\tau)) \in \mathbb{C} \times \mathbb{R},$$

is called *horizontal* if  $\dot{\gamma}(\tau) \in H_{\gamma(\tau)}(\mathfrak{H})$  for almost all  $\tau \in [a, b]$ ; equivalently,

$$\dot{t}(\tau) = -2\Im \left( \overline{z(\tau)} \dot{z}(\tau) \right),$$

for almost all  $\tau \in [a, b]$ . A curve  $\gamma : [a, b] \rightarrow \mathfrak{H}$  is absolutely continuous with respect to  $d_{\mathfrak{H}}$  if and only if it is a horizontal curve. Moreover, the horizontal length of a smooth rectifiable curve  $\gamma = (\gamma_h, \gamma_3)$  with respect to  $\|\cdot\|$  is given by the integral over the (Euclidean) norm of the horizontal part of the tangent vector,

$$\ell_h(\gamma) = \int_a^b \|\dot{\gamma}_h(\tau)\| d\tau = \int_a^b \left( \langle \dot{\gamma}(\tau), X_{\gamma(\tau)} \rangle^2 + \langle \dot{\gamma}(\tau), Y_{\gamma(\tau)} \rangle^2 \right)^{1/2} d\tau.$$

Thus the sub-Riemannian or *Carnot-Carathéodory distance* of two arbitrary points  $p, q \in \mathfrak{H}$  is

$$d_{cc}(p, q) = \inf_{\gamma} \ell_h(\gamma),$$

where  $\gamma$  is horizontal and joins  $p$  and  $q$  (and therefore horizontal curves are *geodesics* for the sub-Riemannian metric). The  $d_{cc}$ -sphere of radius  $R$  and centered at the origin is the *Carnot-Carathéodory sphere*  $\mathcal{S}_{cc}(R)$  and is constructed as follows. Consider the family  $\mathbf{c}_k$ ,  $k \in \mathbb{R}$ , of planar circles where

$$\mathbf{c}_k(s) = \frac{1}{k}(1 - e^{iks}), \quad s \in [0, 2\pi/|k|].$$

In the case where  $k = 0$ , circles degenerate into straight line segments. For every  $k$ ,  $\mathbf{c}_k$  is lifted to the horizontal curve  $\mathbf{p}_0^k$  where

$$\mathbf{p}_0^k(s) = (\mathbf{c}_k(s), t_k(s)), \quad t_k(s) = \frac{2}{k} \left( \frac{1}{k} \sin(ks) - s \right).$$

Denote by  $\mathbf{p}_{\phi}^k$  the rotation of  $\mathbf{p}_0^k$  around the vertical axis  $\mathcal{V} = \{(z, t) \in \mathfrak{H} \mid z = 0\}$ . Then

$$\mathcal{S}_{cc}(R) = \mathbf{p}_{\phi}^k(R), \quad k \in [-2\pi/R, 2\pi/R], \quad \phi \in [0, 2\pi].$$

In comparison, the Korányi-Cygan metric and the Carnot-Carathéodory metric are equivalent metrics, they both behave like the Euclidean metric in horizontal directions ( $X$  and  $Y$ ), and behave like the square root of the Euclidean metric in the missing direction ( $T$ ). Their isometry groups are the same, but not their similarity groups: inversion is not a similarity of  $d_{cc}$ . The relation of  $d_{\mathfrak{H}}$  (which is not a geodesic metric) and  $d_{cc}$  is given as follows, see [9]: there exist constants  $C_1, C_2 > 0$  so that

$$(1) \quad C_1 d_{\mathfrak{H}}(p, 0) \leq d_{cc}(p, 0) \leq C_2 d_{\mathfrak{H}}(p, 0),$$

for each  $p \in \mathfrak{H}$ . Finally,  $d_{\mathfrak{H}}$  and  $d_{cc}$  generate the same infinitesimal structure: If  $\gamma : [0, 1] \rightarrow \mathfrak{H}$  is a  $\mathcal{C}^1$  curve and  $t_i = i/n$ ,  $i = 1, \dots, n$ , is a partition of  $[0, 1]$ , then

$$\limsup_{n \rightarrow \infty} \sum_{i=1}^n d_{\mathfrak{H}}(\gamma(t_i), \gamma(t_{i-1})) = \begin{cases} \ell(\gamma) & \text{if } \gamma \text{ is horizontal,} \\ \infty & \text{otherwise.} \end{cases}$$

Here,  $\ell(\gamma)$  denotes the length of  $\gamma$  with respect to both  $d_{\mathfrak{H}}$  and  $d_{cc}$ , see Lemma 2.4 in [9].

Contact transformations between domains (open and connected subsets) of  $\mathfrak{H}$  play an important role in the theory of quasiconformal mappings of  $\mathfrak{H}$ . A *contact transformation*  $f : \Omega \rightarrow \Omega'$  on  $\mathfrak{H}$  is a diffeomorphism between domains  $\Omega$  and  $\Omega'$  in  $\mathfrak{H}$  which preserves the contact structure, i.e.,

$$(2) \quad f^* \omega = \lambda \omega,$$

for some non-vanishing real valued function  $\lambda$ . We write  $f = (f_I, f_3)$ ,  $f_I = f_1 + if_2$ . Then a contact mapping  $f$  is completely determined by  $f_I$  in the sense that the contact condition (2) is equivalent to the following system of partial differential equations:

$$\begin{aligned} (3) \quad & \bar{f}_I Z f_I - f_I Z \bar{f}_I + i Z f_3 = 0, \\ (4) \quad & f_I \bar{Z} \bar{f}_I - \bar{f}_I \bar{Z} f_I - i \bar{Z} f_3 = 0, \\ (5) \quad & -i(\bar{f}_I T f_I - f_I T \bar{f}_I + i T f_3) = \lambda. \end{aligned}$$

If  $f$  is  $\mathcal{C}^2$  it is elementary to prove that

$$\det J_f = \lambda^2,$$

where by  $J_f$  we denote the usual Jacobian matrix of  $f$ .

### 3. A BRIEF OVERVIEW OF THE KORÁNYI-REIMANN THEORY

We start by recalling various definitions of quasiconformal mappings on the Heisenberg group  $\mathfrak{H}$ . All these definitions turn out to be equivalent. In our exposition, we follow the lines of [22] and [23] with minor deviations.

*Metric definition.* Korányi and Reimann wrote in [22]: *The (metric) definition is a straightforward generalisation of the corresponding notion in Euclidean space. It was Mostow who for the first time studied these mappings in the context of semisimple Lie groups of rank one.* We consider the Heisenberg group  $\mathfrak{H}$  equipped with its metric  $d_{\mathfrak{H}}$ . For a homeomorphism  $f$  between domains  $\Omega$  and  $\Omega'$  in  $\mathfrak{H}$  and for  $p \in \Omega$  we set

$$\begin{aligned} L_f(p, r) &= \sup_{d_{\mathfrak{H}}(p, q) \leq r, q \in \Omega} d_{\mathfrak{H}}(f(p), f(q)), \\ l_f(p, r) &= \inf_{d_{\mathfrak{H}}(p, q) \geq r, q \in \Omega} d_{\mathfrak{H}}(f(p), f(q)), \\ H_f(p, r) &= \frac{L(p, r)}{l(p, r)} \end{aligned}$$

and

$$H_f(p) = \limsup_{r \rightarrow 0} H(p, r).$$

**Definition 3.1. (Metric definition)** A homeomorphism  $f : \Omega \rightarrow \Omega'$  between domains  $\Omega$  and  $\Omega'$  in  $\mathfrak{H}$  is quasiconformal if it is uniformly bounded from above in  $\Omega$ . If in addition

$$\text{ess sup}_{p \in \Omega} H_f(p) = \|H_f\|_{\infty} \leq K,$$

then  $f$  is called  $K$ -quasiconformal.

We note that due to (1), we obtain an equivalent definition if we substitute  $d_{\mathfrak{H}}$  by the metric  $d_{cc}$ , see for instance the metric definition in [23]. Moreover, conformal mappings (i.e., elements of  $\text{SU}(2, 1)$ , the isometry group of complex hyperbolic plane  $\mathbf{H}_{\mathbb{C}}^2$ , acting on  $\mathfrak{H}$ ) are 1-quasiconformal. The converse is also true; a 1-quasiconformal mapping is necessarily an element of  $\text{SU}(2, 1)$ , see [22] for a proof.

In the context of the Heisenberg group, there are various equivalent analytic definitions of quasiconformal mappings  $f : \Omega \rightarrow \mathfrak{H}$  which are all equivalent to the metric definition 3.1. We refer the reader to [11], [17] and [39]. For the definition we are about to state, see [4] and the references therein.

*Analytic definition.* We first define  $HW^{1,4}(\Omega, \mathfrak{H})$ , the *horizontal Sobolev space*. We say that a function  $u : \Omega \rightarrow \mathbb{C}$  is in  $HW^{1,4}(\Omega, \mathfrak{H})$  if it is in  $L^4(\Omega, \mathfrak{H})$  and if there exist functions  $v, w \in L^4(\Omega, \mathfrak{H})$  such that

$$\int_{\Omega} v \phi d\mathcal{L}^3 = - \int_{\Omega} u Z \phi d\mathcal{L}^3 \quad \text{and} \quad \int_{\Omega} w \phi d\mathcal{L}^3 = - \int_{\Omega} u \bar{Z} \phi d\mathcal{L}^3$$

for all  $\phi \in \mathcal{C}_0^\infty(\Omega, \mathbb{C})$ . Now a mapping  $f : \Omega \rightarrow \mathfrak{H}$ ,  $f = (f_I, f_3)$  is said to be in  $HW^{1,4}(\Omega, \mathfrak{H})$  if both  $f_I, f_3$  are in  $HW^{1,4}(\Omega, \mathfrak{H})$ . Such a mapping which also satisfies Conditions (3) and (4) a.e. is called *weakly contact* and one can define its formal horizontal differential  $(D_h)f_p$  at almost all  $p$ , which in matrix form is given by

$$(D_h)f_p = \begin{pmatrix} Z f_I & \bar{Z} f_I \\ Z \bar{f}_I & \bar{Z} \bar{f}_I \end{pmatrix}_p.$$

This is extended to a Lie algebra homomorphism known as the *Pansu derivative*  $(D_0)f_p$ , see [32], which in matrix form is

$$(D_0)f_p = \begin{pmatrix} Z f_I & \bar{Z} f_I & 0 \\ Z \bar{f}_I & \bar{Z} \bar{f}_I & 0 \\ 0 & 0 & |Z f_I|^2 - |\bar{Z} f_I|^2 \end{pmatrix}_p.$$

It is worth noticing that Pansu derivation is the Heisenberg analogue of the plain derivation in Euclidean spaces; a mapping  $f : \Omega \rightarrow \Omega'$  between domains of  $\mathfrak{H}$  is  $P$ -differentiable at  $p \in \Omega$  if for  $c \rightarrow 0$  the mappings

$$D_c^{-1} \circ T_{f(p)}^{-1} \circ f \circ T_{f(p)} \circ D_c$$

converge locally uniformly to a homomorphism  $(D_0)f_p$  from  $T_p(\mathfrak{H})$  to  $T_{f(p)}(\mathfrak{H})$  which preserves the horizontal space  $H(\mathfrak{H})$ . Here  $D$  and  $T$  are dilations and left translations respectively.

Now let

- (1)  $\|(D_h)f_p\| := \max \{ \|(D_h)f_p(V)\| \mid \|V\| = 1 \} = |Z f_I(p)| + |\bar{Z} f_I(p)|$  a.e.;
- (2)  $J_f(p) = \det(D_0)f_p = (\det(D_h)f_p)^2 = (|Z f_I(p)|^2 - |\bar{Z} f_I(p)|^2)^2$ ;
- (3)

$$K(p, f)^2 = \frac{\|(D_h)f_p\|^4}{J_f(p)} = \left( \frac{|Z f_I(p)| + |\bar{Z} f_I(p)|}{|Z f_I(p)| - |\bar{Z} f_I(p)|} \right)^2.$$

**Definition 3.2. (Analytic definition)** A homeomorphism  $f : \Omega \rightarrow \Omega'$ ,  $f = (f_I, f_3)$ , between domains in  $\mathfrak{H}$  is an orientation-preserving  $K$ -quasiconformal mapping if  $f \in HW^{1,4}(\Omega, \mathfrak{H})$  is weakly contact and if

$$\|(D_h)f_p\|^4 \leq K J_f(p),$$

for almost all  $p$ .

The function  $\Omega \ni p \mapsto K(p, f) \in [1, \infty)$  is the *distortion function* of  $f$  and the constant of quasiconformality  $K$  is also called the *maximal distortion* of  $f$ .

We remark that the analytic definition given by Korányi and Reimann, see for instance [23], is based on the notion of absolute continuity on lines (ACL): Mappings with this property are absolutely continuous on a.e. fiber of any smooth fibration determined by a horizontal vector field  $V$ . In the case of the Heisenberg group  $\mathfrak{H}$ , absolute continuity holds on almost all fibers of smooth *horizontal* fibrations. For such a fibration, the fibers  $\gamma_p$  can be parametrised by the flow  $f_s$  of a horizontal *unit* vector field  $V$ : i.e.,  $V$  is of the form  $aX + bY$  with  $a^2 + b^2 = 1$ . Mostow proved (Theorem A in [23]) that quasiconformal mappings are absolutely continuous



on a.e. fiber  $\gamma$  of any given fibration  $\Gamma_V$  determined by a left invariant horizontal vector field  $V$ . The Euclidean counterpart was proved by Gehring in [15]. The proof in the Euclidean case is considerably easier. In this way, Korányi and Reimann defined a homeomorphism  $f : \Omega \rightarrow \Omega'$ ,  $f = (f_I, f_3)$ , between domains in  $\mathfrak{H}$  to be an orientation-preserving  $K$ -quasiconformal mapping if

- (i) it is ACL;
- (ii) it is a.e.  $P$ -differentiable, and
- (iii) it satisfies a.e. the system of Beltrami equations

$$(6) \quad \overline{Z}f_I = \mu Zf_I,$$

$$(7) \quad \overline{Z}f_{II} = \mu Zf_{II},$$

where  $f_{II} = f_3 + i|f_I|^2$  and  $\mu$  is a complex function in  $\Omega$  such that

$$\frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty} \leq K \quad \text{a.e.}$$

Here,  $\|\mu\|_\infty = \text{esssup}\{|\mu(z, t)| \mid (z, t) \in \Omega\}$ . For each  $p = (z, t) \in \Omega$ , the function

$$\mu(p) = \mu_f(p) = \frac{\overline{Z}f_I(p)}{Zf_I(p)}$$

is called the *Beltrami coefficient* of  $f$ .

*Geometric definition.* Korányi and Reimann proved the equivalence of the metric and analytic definitions by showing that both are equivalent to a third definition which they called the geometric definition. This involves the notion of *capacity* of a ring domain  $\text{Cap}(E, F)$  in  $\mathfrak{H}$ : by  $(E, G)$  we denote the open bounded subset  $U = \mathfrak{H} \setminus (E \cup F)$  where  $E$  and  $F$  are disjoint connected closed subsets of  $\mathfrak{H}$  and moreover  $E$  is compact. Then the capacity of  $(E, G)$  is

$$\text{Cap}(E, F) = \inf \int_{\mathfrak{H}} |\nabla_h u|^4 d\mathcal{L}^3,$$

where the infimum is taken over all smooth functions  $u$  in  $G$  with  $u|_E \geq 1$  and  $u|_F = 0$ . Here  $\nabla_h u$  is the horizontal gradient  $(Xu)X + (Yu)Y$ , and  $d\mathcal{L}^3$  is the volume element of the usual Lebesgue measure in  $\mathbb{C} \times \mathbb{R}$ . For details, see Section 3 in [23].

**Definition 3.3. (Geometric definition I)** Let  $f : \Omega \rightarrow \Omega'$  be a homeomorphism between domains in  $\mathfrak{H}$ . Then  $f$  is quasiconformal if there exists a  $K \geq 1$  such that for each ring domain  $(E, F)$  in  $\Omega$  we have the capacity inequality

$$\text{Cap}(E, F) \leq K^2 \text{Cap}f(E, F).$$

In what follows we will give another geometric definition which involves the notion of *modulus* of families of curves. Let  $\Gamma$  be a family of rectifiable (with respect to  $d_{\mathfrak{H}}$ ) curves lying in a domain  $\Omega \subset \mathfrak{H}$ , i.e., the curves have finite length with respect to  $d_{\mathfrak{H}}$  (or in other words, they have finite horizontal length). If  $\rho : \mathfrak{H} \rightarrow [0, \infty)$  is a non-negative Borel function and  $\gamma$  is a parametrisation of a rectifiable curve  $\gamma(t) = (\gamma_h(t), \gamma_3(t))$ ,  $t \in [a, b]$ , we define

$$\int_{\gamma} \rho ds = \int_a^b \rho(\gamma(t)) \|\dot{\gamma}_h(t)\| dt.$$



Let  $\text{adm}(\Gamma)$  be the set of these non-negative Borel functions  $\rho$  defined in  $\mathfrak{H}$  which satisfy

$$\int_{\gamma} \rho ds \geq 1, \quad \text{for all rectifiable } \gamma \in \Gamma.$$

Then the *modulus*  $\text{Mod}(\Gamma)$  of  $\Gamma$  is defined as

$$\text{Mod}(\Gamma) = \inf_{\rho \in \text{adm}(\Gamma)} \int_{\mathfrak{H}} \rho^4(p) d\mathcal{L}^3(p),$$

where  $d\mathcal{L}^3$  is the volume element of the usual Lebesgue measure in  $\mathbb{C} \times \mathbb{R}$ .

A fundamental inequality which may be found for instance in [4], is formulated as follows. Let  $f : \Omega \rightarrow \Omega'$  be a  $K$ -quasiconformal mapping between domains in  $\mathfrak{H}$  and let  $\Gamma$  be a family of curves in  $\Omega$ . Then in the first place we have

$$(8) \quad \text{Mod}(f(\Gamma)) \leq \int_{\Omega} K(p, f)^2 \rho^4(p) d\mathcal{L}^3(p).$$

Here,  $K(p, f)$  is the distortion function of  $f$  and the integral on the right is called the *distortion functional* of  $f$ . From (8) we then obtain

$$(9) \quad \frac{1}{K^2} \text{Mod}(\Gamma) \leq \text{Mod}(f(\Gamma)) \leq K^2 \text{Mod}(\Gamma).$$

Inequality (9) is known as the *modulus inequality*. As a direct corollary of the modulus inequality we obtain that the modulus of a family of curves is a conformal invariant; if  $f$  is conformal then  $\text{Mod}(f(\Gamma)) = \text{Mod}(\Gamma)$ . Now the second geometric definition of quasiconformality stands as follows.

**Definition 3.4. (Geometric definition II)** Let  $f : \Omega \rightarrow \Omega'$  be a homeomorphism between domains in  $\mathfrak{H}$ . Then  $f$  is quasiconformal if there exists a  $K \geq 1$  such that for each curve family  $\Gamma$  in  $\Omega$  the modulus inequality (9) holds.

Since the capacity of a ring domain  $(E, G)$  is equal to the modulus of the family of horizontal curves joining  $E$  and  $F$  in  $U$ , see [13] or Proposition 2.4 of [19], the geometric definition II implies the geometric definition I. The converse may be derived via *quasisymmetric mappings*. Like in the classical case, quasiconformal mappings are strongly related to quasisymmetric ones. A mapping  $f : \Omega \rightarrow \Omega'$  between domains of  $\mathfrak{H}$  is called *locally  $\eta$ -quasisymmetric* if there exists an increasing self-homeomorphism  $\eta$  of  $[0, \infty)$  such that for each Whitney ball  $B \subset \Omega$ ,

$$\frac{d_{\mathfrak{H}}(f(p), f(q))}{d_{\mathfrak{H}}(f(p), f(r))} \leq \eta \left( \frac{d_{\mathfrak{H}}(p, q)}{d_{\mathfrak{H}}(p, r)} \right),$$

for all  $p, q, r \in B$ ,  $p \neq r$ . Recall that a Whitney ball  $B \subset \Omega$  is a closed metric ball  $B(x, R)$  centered at  $x \in \Omega$  and with radius  $R$ , such that  $2B = B(x, 2R) \subset \Omega$ .

The equivalence of all the afore stated definitions of quasiconformality is clarified by a theorem which in its full generality is Theorem 9.8 in [21], see also Theorem 6.33 in [9]. This result states that if  $f : \Omega \rightarrow \Omega'$  is a homeomorphism between domains of  $\mathfrak{H}$ , then the following are equivalent.

- i)  $f$  is quasiconformal according to the metric definition 3.1;
- ii)  $f$  is locally  $\eta$ -quasisymmetric;
- iii)  $f$  is quasiconformal according to the geometric definition 3.4.

In this manner, it follows that i), ii) and iii) are all equivalent to the analytic definition 3.2 and to geometric definition 3.3 as well.

*Smooth quasiconformal mappings.* In their first paper [22], Korányi and Reimann studied mostly smooth quasiconformal mappings. Quasiconformal mappings with sufficient smoothness have to be contact transformations. This property distinguishes quasiconformal mappings on the Heisenberg group  $\mathfrak{H}$  from those defined on Euclidean spaces. In fact, from  $P$ -differentiability of quasiconformal mappings it follows that  $P$ -diffeomorphic  $K$ -quasiconformal mappings are contact transformations satisfying

$$(10) \quad \|(D_h)f\|^4 \leq K|J_f| \quad \text{a.e.}$$

(Here, the absolute value in the Jacobian covers both situations of orientation-preserving and orientation-reversing mappings). The converse is also true, see Proposition 8 in [23]: if a  $\mathbb{C}^2$  contact transformation  $f$  satisfies Condition (10), then  $f$  is  $K$ -quasiconformal. We conclude that  $K$ -quasiconformal diffeomorphisms lie in the class of contact transformations. Due to the contact Conditions (3), (4) and (5), Equation (6) in the Beltrami system implies Equation (7).

*Quasiconformal deformations and measurable Riemann Mapping Theorem.* Perhaps one of the most striking results in the original work of Korányi and Reimann, is their generalisation to the Heisenberg setting of the famous measurable Riemann Mapping Theorem in its infinitesimal version. Besides its genuine importance, this theorem enables us to construct as many quasiconformal mappings on  $\mathfrak{H}$  as we wish, out of *quasiconformal deformations*. Let  $f_s : \mathfrak{H} \rightarrow \mathfrak{H}$ ,  $f_s = f_s(z, t)$ ,  $s \in \mathbb{R}$ , be a  $\mathbb{C}^1$  one-parameter group of transformations of  $\mathfrak{H}$  with infinitesimal generator  $V$ , satisfying the initial condition  $f_0(z, t) = \text{id}$ . Then the following differential equation holds:

$$\frac{d}{ds} f_s(z, t) = V(f_s(z, t)).$$

We are interested primarily in one-parameter groups of *contact* transformations since we have seen that smooth enough quasiconformal mappings are contact. Infinitesimal generators of one-parameter groups of contact transformations have been studied by Libermann and are of a special form: According to Theorem 5 in [22],  $C^1$  vector fields of the form

$$(11) \quad V = -\frac{1}{4} [(Yp)X - X(p)Y] + pT = \frac{i}{2} [(\overline{Z}p)Z - (Zp)\overline{Z}] + pT,$$

where  $p$  is an arbitrary real valued function, generate local one-parameter groups of contact transformations. Conversely, every  $\mathbb{C}^1$  vector field  $V$  which generates a local one-parameter group of contact transformations is necessarily of this form with  $p = \omega(V)$ .

A precise estimate for the constant of quasiconformality of a one-parameter group of quasiconformal mappings generated by a  $\mathbb{C}^2$  vector field is given by Theorem 6 in [22]; this is also the first version of the measurable Riemann mapping theorem for the Heisenberg group case: Suppose that  $V$  is a  $\mathbb{C}^2$  vector field of the form (11) generating a one-parameter group  $\{f_s\}$  of contact transformations. If

$$|ZZp| \leq c^2/2,$$

then  $f_s$  is  $K$ -quasiconformal with the constant of quasiconformality  $K = K(s)$  of  $f_s$  satisfying

$$K + \frac{1}{K} \leq 2e^{c|s|}.$$

Korányi and Reimann improved this result, see Theorem H in [23]: The assumptions there for the vector field  $V$  of the form (11) is to simply be continuous with compact support in  $\mathfrak{H}$ ; as for the derivatives  $ZZp$  it suffices to consider them in their distributional sense.

*Extension of quasiconformal deformations.* The measurable Riemann mapping theorem in the Heisenberg group case is the infinitesimal analogue of the measurable Riemann mapping theorem of Ahlfors and Bers in the Heisenberg setting, but there is no result assuring the existence of a solution to the Beltrami system of Equations (6) and (7). However, this theorem constitutes the key step to pass from quasiconformal deformations of the Heisenberg group  $\mathfrak{H}$  to quasiconformal deformations of the complex hyperbolic plane  $\mathbf{H}_{\mathbb{C}}^2$ . In the following we describe this passage, restricting ourselves to the smooth case. Assuming enough smoothness, quasiconformal mappings of  $\mathbf{H}_{\mathbb{C}}^2$  are necessarily symplectic transformations, i.e., diffeomorphisms  $F$  such that  $F^*\Omega = \Omega$ , where  $\Omega$  is the symplectic form in  $\mathbf{H}_{\mathbb{C}}^2$  derived by its Kähler metric. If  $J$  is the natural complex structure in  $\mathbf{H}_{\mathbb{C}}^2$ , then  $F$  defines another complex structure  $J_\mu = F_*^{-1} \circ J \circ F$  in  $\mathbf{H}_{\mathbb{C}}^2$  and there is an associated complex antilinear self-mapping of the  $(1,0)$ -tangent bundle  $T^{(1,0)}(\mathbf{H}_{\mathbb{C}}^2)$  such that the  $(1,0)$ -tangent bundle of  $J_\mu$  is  $\{Z - \overline{\mu Z} \mid Z \in T^{(1,0)}\}$ . The map  $\mu$  is called the *complex dilation* of  $F$  and there is a description of  $\mu$  via a Beltrami system of equations, see pp. 401–402 in [12]. The connection between quasiconformal symplectic transformations of the complex hyperbolic plane and quasiconformal contact transformations of the Heisenberg group is described as follows, see [24] and [25]:

- (i) A (quasiconformal) symplectic transformation  $F$  of the complex hyperbolic plane  $\mathbf{H}_{\mathbb{C}}^2$  extends to a (quasiconformal) contact transformation of the boundary.
- (ii) A quasiconformal deformation of the boundary extends to a quasiconformal deformation in the interior.

In both cases, the constant of quasiconformality remains the same.

#### 4. FURTHER DEVELOPMENTS AND SOME OPEN PROBLEMS

*Extremal problems.* In the classical theory, *extremal* quasiconformal mappings are the ones minimising the maximal distortion (constant of quasiconformality) within a certain class of mappings in the complex plane or between Riemann surfaces. Since the times of Grötzsch and Teichmüller, a method based on the modulus of curve families has been applied to detect such mappings; it turned out that the very same method applies for the mappings which minimise a mean distortion functional in the class of quasiconformal mappings between annuli in the complex plane, [5]. Given a domain  $\Omega$  in the complex plane, a family  $\mathcal{F}$  of mappings defined in  $\Omega$  and a density  $\rho$  corresponding to the geometry of  $\Omega$ , the *mean distortion functional*  $\mathfrak{M}_1(f, \rho)$  is

$$\mathfrak{M}_1(f, \rho) = \frac{\int_{\Omega} K(p, f) \rho(p)^2 d\mathcal{L}^2}{\int_{\Omega} \rho(p)^2 d\mathcal{L}^2}, \quad f \in \mathcal{F}.$$

Recently in [4], a variation of the modulus method has been developed in the Heisenberg group setting by Balogh, Fässler and Platis, to prove that there exists a minimiser of a mean distortion functional in the class of quasiconformal mappings between Heisenberg spherical annuli. Given a domain  $\Omega$  in the Heisenberg group, a family  $\mathcal{F}$  of mappings defined in  $\Omega$  and a density  $\rho$  corresponding to the geometry of  $\Omega$ , the *mean distortion functional*  $\mathfrak{M}_2(f, \rho)$  is

$$\mathfrak{M}_2(f, \rho) = \frac{\int_{\Omega} K(p, f)^2 \rho(p)^4 d\mathcal{L}^3}{\int_{\Omega} \rho(p)^4 d\mathcal{L}^3}, \quad f \in \mathcal{F}.$$

The minimiser in [4] is the *Heisenberg stretch map*, an extension of the usual stretch map

$$(12) \quad f_k(z) = z|z|^{(1/K)-1}, \quad K > 1,$$

of the plane. Moreover, the Heisenberg stretch map is essentially the unique minimiser of the mean distortion functional, see [6], a fact that is in strong contrast to the classical case where there exist infinite such minimisers, see [8]. However, it does not minimise the maximal distortion and this is also in contrast to the classical situation. The problem of finding such a minimiser is open. We note here that the modulus method is up to now the unique tool for the detection of extremal mappings; in the Heisenberg setting results similar to Teichmüller's Existence and Uniqueness theorems are not available. Moreover, in the application of the modulus method there is an additional difficulty due to the lack of a Riemann Mapping Theorem. In the classical case, one can always reduce an extremal problem concerning mappings defined on a ring domain to an extremal problem about mappings defined in a circular annulus. But in the Heisenberg group case things are rather different: In [27], Korányi and Reimann calculated the capacity (that is, the modulus of the family of curves joining the two components of the boundary) of a ring whose boundary comprises two homocentric Korányi-Cygan spheres. Rather surprisingly, a variety of rings centered at the origin (like rings whose boundary comprises of two Carnot-Carathéodory spheres), have exactly the same capacity, see [33]. In general though, the calculation of moduli of curve families in an arbitrary ring inside  $\mathfrak{H}$  is a difficult task.

Extremal problems also arise naturally in the theory of quasiconformal mappings of compact pseudoconvex CR manifolds. Such mappings have been defined by Korányi and Reimann in [26] and the extremality problem can be stated as follows. Given two CR structures on a 3-dimensional contact manifold, determine the quasiconformal homeomorphisms that have the least maximal distortion with respect to the two CR structures. Problems of this type have been studied by various authors, see for instance the works of Miner [28], and Tang [36].

*Sobolev and Hölder exponents.* Besides being a minimiser of the mean distortion functional and of the maximal distortion, the planar stretch map (12) is extremal for various other problems. Astala showed in [2] that a planar  $K$ -quasiconformal mapping lies in the Sobolev space  $W_{loc}^{1,p}$  with  $p < 2K/(K-1)$ . The example of the stretch map demonstrates that the given bound is sharp and this proves Gehring's conjecture, see [16], on the exponent of higher integrability in the two-dimensional case. Ahlfors showed in [1] that a planar  $K$ -quasiconformal mapping is locally Hölder continuous with exponent  $1/K$ ; again the stretch map can be used to show that this exponent cannot be improved.

The analogue of Gehring's higher integrability result for quasiconformal mappings on the Heisenberg group has been established by Korányi-Reimann in [24]: A quasiconformal mapping  $f$  on  $\mathfrak{H}$  lies in  $HW_{loc}^{1,p}(\Omega, \mathfrak{H})$  for an exponent  $p > 4$ . However, up to present there is no sharp upper bound known in the spirit of Astala. Using the Heisenberg stretch mapping, it is shown in [4] that

$$p(\mathfrak{H}, K) \leq \frac{4K^{\frac{1}{4}}}{K^{\frac{1}{4}} - 1},$$

where

$$p(\mathfrak{H}, K) = \sup \left\{ p \geq 1 : f \in HW_{loc}^{1,p}(\mathfrak{H}, \mathfrak{H}), f : \mathfrak{H} \rightarrow \mathfrak{H} \text{ } K\text{-quasiconformal} \right\}.$$

On the Hölder continuity side, it is known (see p. 53 in [24]) that quasiconformal mappings on the Heisenberg group and on more general Carnot groups (see [17]) are locally Hölder continuous. A bound for the Hölder exponent in terms of the distortion has been given in

Theorem 6.6 in [3] and it is not likely that we can use the Heisenberg stretch to prove that this bound is optimal. Therefore, what is the appropriate quasiconformal mapping?

*Quasiconformal maps in abstract spaces.* The study of quasiconformal mappings in the Heisenberg group  $\mathfrak{H}$  motivated the study of quasiconformal mappings to larger and more abstract spaces; some of which are CR manifolds, metric spaces with controlled geometry, Carnot groups and sub-Riemannian (Carnot-Carathéodory) spaces. It also gave rise to questions concerning the comparison between the classical Ahlfors-Bers and the Korányi-Reimann theory. It is equally fascinating to detect the points where similarities do exist, but also the points where they break down; all these are briefly addressed below. We start from spaces with controlled geometry, Loewner spaces and Carnot groups. We refer the reader to the pioneering work of Heinonen and Koskela [19] and [20], as well as to the notes of Reimann [35]. In general, metric spaces with controlled geometry are the metric spaces which display some kind of regularity with respect to comparison of distance and volume; the latter is the essence of quasiconformal mappings. Such a metric space  $(X, d)$  of dimension  $Q > 1$  (also known as an *Ahlfors-David regular metric space*) is endowed with a Borel measure  $\mu$  compatible with the metric  $d$  in the following way: there exists a constant  $C \geq 1$  such that for all metric balls  $B_R$  of radius  $R < \text{diam}(X)$  the following inequality holds:

$$\frac{1}{C}R^Q \leq \mu(B_R) \leq CR^Q.$$

Quasiconformal mappings are defined in such spaces via the metric definition, and the same holds for notions like the modulus of curve families and quasisymmetric mappings. There are well defined notions of  $Q$ -modulus  $\text{Mod}_Q(\Gamma)$  of a family of curves  $\Gamma$  and of quasisymmetry, entirely analogous to those holding in the Heisenberg setting.

A metric space  $(X, d)$  is called a  *$Q$ -Loewner space*, if there is a strictly increasing self-mapping  $\eta$  of  $(0, \infty)$  such that  $\text{Mod}_Q(\Gamma) \geq \eta(k)$ , where  $\Gamma$  is the family of curves connecting two continua  $C_0$  and  $C_1$  with

$$\min\{\text{diam}(C_0), \text{diam}(C_1)\} \geq k \text{dist}(C_0, C_1).$$

The Heisenberg group  $\mathfrak{H}$  endowed with the Carnot-Carathéodory metric  $d_{cc}$  is a 3-regular Loewner space, and a bigger class of Loewner spaces are the so-called *Carnot groups*. A Carnot group is a simply connected nilpotent Lie group  $N$  with a derivation  $\alpha$  on its Lie algebra  $\mathfrak{n}$  such that  $\ker(\alpha)$  generates  $\mathfrak{n}$ . Via the exponential map,  $N$  and subsequently  $\mathfrak{n}$  are identified to  $\mathbb{R}^m$  for some  $m \in \mathbb{N}$  and the group action is given by the Campbell-Hausdorff formula, see [9]. The Haar measure of  $N$  is just the Lebesgue measure of  $\mathbb{R}^m$  and a Carnot-Carathéodory distance is well defined. The starting point of the study of Carnot groups is probably Pansu's thesis [32], see also [31]. A quite extensive study of the topic of quasiconformal analysis of Carnot groups is found in the work of Vodop'yanov, see for instance [39], [40] and [41].

The primary problem that one is facing in the study of the above spaces is to give proper analytic and geometric definitions of quasiconformality which are equivalent to the general (and applying in all cases) metric definition. In this direction, see the works of Williams [42], Tyson [37] and [38], and Heinonen et al. [21]. On the other hand, the conditions of the metric definition itself can be substantially relaxed and this gives rise to quite striking results, see [7] and the bibliography given there.

*The holy grail.* In contrast to the Teichmüller space case where extremal quasiconformal mappings are used to describe the whole space, it seems that a lot of effort has to be made

to obtain (or not!) an analogous result for spaces like the complex hyperbolic quasi-Fuchsian space which is defined now. Complex hyperbolic quasi-Fuchsian space  $\mathcal{Q}_{\mathbb{C}}(\Sigma)$  of a closed surface  $\Sigma$  of genus  $g > 1$  is perhaps the most natural extension of the Teichmüller space of  $\Sigma$ : it consists of representations of the fundamental group  $\pi_1(\Sigma)$  into the isometry group  $SU(2, 1)$  of complex hyperbolic plane which are discrete, faithful, totally loxodromic and geometrically finite. We underline here that those conditions prevent that space (as well as the real quasi-Fuchsian space, that is the space of discrete, faithful, totally loxodromic and geometrically finite representations of  $\pi_1(\Sigma)$  into  $PSL(2, \mathbb{C})$ ) to fall in the Mostow rigidity setting; the representations are convex cocompact and not cocompact as in the assumptions of Mostow's rigidity theorem. In a convex cocompact representation the quotient of the convex hull of the limit set has finite volume (so it may have infinite funnel ends but no cusps). Thus the limit set can never be the entire boundary; there is always a region of discontinuity. In particular, for quasi-Fuchsian or complex hyperbolic quasi-Fuchsian groups the limit set is a topological circle and there is a domain of discontinuity in the boundary.

There is a quite large bibliography on the subject. For a summary of results concerning  $\mathcal{Q}_{\mathbb{C}}(\Sigma)$  we refer the reader to [34]. Perhaps the most prominent problem in the subject is to examine the analytical structure of  $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ . In the case of Teichmüller space this is carried out via the Ahlfors-Bers theory and the challenge here is to use the Korányi-Reimann theory of quasiconformal mappings in the Heisenberg group to obtain similar results. In this direction, and regardless of the lack of an existence theorem for the solution of the Beltrami equation, one is invited to start from an irreducible representation  $\rho \in \mathcal{Q}_{\mathbb{C}}(\Sigma)$  and to construct quasiconformal deformations on the Heisenberg group with starting point  $\rho$ , to determine exactly the tangent space at  $\rho$  from the vector fields generating these deformations. The problem is still open (it has been named *the holy grail* by the researchers in the area).

## REFERENCES

- [1] L. V. Ahlfors, On quasiconformal mappings. *J. Anal. Math.* 3 (1954), 1–58; correction 207–208.
- [2] K. Astala, Area distortion of quasiconformal mappings. *Acta Math.* 173 (1994), 37–60.
- [3] Z. M. Balogh, I. Holopainen, and J. T. Tyson, Singular solutions, homogeneous norms, and quasiconformal mappings in Carnot groups. *Math. Ann.* 324 (2002), 159–186.
- [4] Z. M. Balogh, K. Fässler, and I. D. Platis, Modulus method and radial stretch map in the Heisenberg group. *Ann. Acad. Sci. Fenn.* 38 (2013), 149–180.
- [5] Z. M. Balogh, K. Fässler, and I. D. Platis, Modulus of curve families and extremality of spiral-stretch maps. *J. Anal. Math.* 113 (2011), 265–291.
- [6] Z. M. Balogh, K. Fässler, and I. D. Platis, Uniqueness of minimisers for a Groetzsch-Belinskii type inequality in the Heisenberg group. *Conf. Geom. and Dyn.* 19 (2015), 122–145.
- [7] Z. M. Balogh, P. Koskela, and S. Rogovin, Absolute continuity of quasiconformal mappings on curves. *Geom. Funct. Anal.* 17 (2007), 645–664.
- [8] P. P. Belinski, *Obschie svoistva kvazikonformnykh otobazhenii*. Izdat. Nauka, Sibirsk. Otdel., Novosibirsk, 1974, 98pp.
- [9] L. Capogna, D. Danielli, S. Pauls, and J. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*. Progress in Mathematics 259. Birkhäuser Verlag, Basel, 2007.
- [10] M. Bourdon, Structure conforme au bord et flot géodésique d'un CAT(-1)-espace. *Enseign. Math.* 41 (1995), 63–102.
- [11] N. S. Dairbekov, On mappings of bounded distortion on the Heisenberg group. *Sib. Math. Zh.* 41 (2000), 49–59.
- [12] S. Dragomir and G. Tomassini, *Differential Geometry and Analysis on CR manifolds*. Progress in Mathematics 246, Birkhäuser Verlag, Berlin, 2006.
- [13] R. Eichmann, *Variationsprobleme auf der Heisenberggruppe*. Lizentiatsarbeit, Universität Bern, Bern, 1990.



- [14] W. Goldman, *Complex hyperbolic geometry*. Oxford Mathematical Monographs. Oxford University Press, New York, 1999.
- [15] F. W. Gehring, The definitions and exceptional sets for quasiconformal mappings. *Ann. Acad. Sci. Fenn.* 281 (1960), 1–28.
- [16] F. W. Gehring, The  $L^p$ -integrability of the partial derivatives of a quasiconformal mapping. *Acta Math.* 130 (1973), 265–277.
- [17] J. Heinonen, Calculus on Carnot groups. In *Fall school in Analysis Jyväskylä, 1994*, Report 68, Univ. Jyväskylä, Jyväskylä, (1995), 1–31.
- [18] J. Heinonen, What is a quasiconformal mapping?. *Not. AMS.* 53 (2006), no. 11, 1334–1335.
- [19] J. Heinonen and P. Koskela, Definitions of quasiconformality. *Invent. Mat.* 120, (1995), 61–79.
- [20] J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.* 181 (1998), 1–61.
- [21] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, Sobolev classes of Banach space-valued functions and quasiconformal mappings. *J. Anal. Mat.* 85 (2001), 87–139.
- [22] A. Korányi and H. M. Reimann, Quasiconformal mappings on the Heisenberg groups. *Invent. Math.* 80 (1985), 309–338.
- [23] A. Korányi and H. M. Reimann, Foundations for the theory of quasiconformal mappings on the Heisenberg groups. *Adv. in Math.* 111 (1995), 1–87.
- [24] A. Korányi and H. M. Reimann, Contact transformations as limits of symplectomorphisms. *C. R. Acad. Sci. Par.* 318 (1994), 1119–1124.
- [25] A. Korányi and H. M. Reimann, Equivariant extension of quasiconformal deformations into the complex unit ball. *Ind. Univ. Math. J.* 47 (1998), 153–176.
- [26] A. Korányi and H. M. Reimann, Quasiconformal mappings on CR manifolds. In *Conference in honour of E. Vesentini*. Springer Verlag Notes 1422, 59–75, Berlin–Heidelberg–New York, Springer 1988.
- [27] A. Korányi and H. M. Reimann, Horizontal normal vectors and conformal capacity of spherical rings in the Heisenberg group. *Bull. Sci. Math.* (2) 111 (1987), 3–21.
- [28] R. R. Miner, Quasiconformal equivalence of spherical CR manifolds. *Ann. Acad. Sci. Fenn. Ser. A. I. Math* 19 (1994), 83–93.
- [29] G. D. Mostow, Quasi-conformal mappings in  $n$ -space and the rigidity of hyperbolic space forms. *Inst. Hautes Études Sci. Publ. Math.* 34 (1968), 53–104.
- [30] G. D. Mostow, Strong rigidity of locally symmetric spaces. *Ann. Math. Stud.* 78 Princeton University Press, Princeton, NJ., 1973.
- [31] G. A. Margulis and G. D. Mostow, The differential of a quasi-conformal mapping of a Carnot-Carathéodory space. *Geom. Funct. Anal.* 5 (1995), 402–433.
- [32] P. Pansu, Métriques de Carnot–Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math.* 129 (1989), 1–60.
- [33] I. D. Platis, Modulus of revolution rings in the Heisenberg group. Preprint, 2015; arXiv:1504:05099 [math:MG].
- [34] J. R. Parker and I. D. Platis, Complex hyperbolic Quasi-Fuchsian groups. In *Geometry of Riemann surfaces*, London Math. Soc. Lecture Note Ser., 368, Cambridge Univ. Press, Cambridge, 2010, 309–355.
- [35] H. M. Reimann, *Quasiconformal mappings on the Heisenberg group*. Lecture Notes, Trento, 2001.
- [36] P. Tang, Quasiconformal homeomorphisms on CR 3-manifolds with symmetries. *Math. Z.* 219 (1995), 49–69.
- [37] J. Tyson, Quasiconformality and quasisymmetry in metric measure spaces. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* 23 (1998), 525–548.
- [38] J. Tyson, Metric and geometric quasiconformality in Ahlfors regular Loewner spaces. *Conf. Geom. Dynam.* 5 (2001), 21–73.
- [39] S. K. Vodop’yanov, Monotone functions and quasiconformal mappings on Carnot groups. *Sib. Math. Zh.* 37 Issue 6, (1996), 1269–1295.
- [40] S. K. Vodop’yanov and N. A. Evseev, Isomorphisms of Sobolev spaces on Carnot groups and quasi-isometric mappings. (Russian) *Sibirsk. Mat. Zh.* 55 (2014), 1001–1039; translation in *Sib. Math. J.* 55 (2014), 817–848.
- [41] S. K. Vodop’yanov, The geometry of Carnot-Carathéodory spaces, quasiconformal analysis, and geometric measure theory. (Russian) *Vladikavkaz. Mat. Zh.* 5 (2003), 14–34 (electronic).



- [42] M. Williams, Geometric and analytic quasiconformality in metric measure spaces. *Proc. Amer. Math. Soc.* 140 (2012), 1251–1266.

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF CRETE, UNIVERSITY CAMPUS, VOUTES, 70013 HERAKLION CRETE, GREECE, EMAIL: [jplatis@math.uoc.gr](mailto:jplatis@math.uoc.gr)  
*E-mail address:* [jplatis@math.uoc.gr](mailto:jplatis@math.uoc.gr)